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Lagrangian Structure of the Two-Dimensional Lotka–Volterra System

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The two-dimensional Lotka-Volterra system exhibits a Langrangian structure. The Lagrangian function is linear in the velocities and consequently singular, but the corresponding dynamics is unique. The methods of Langrangian mechanics can be applied to this system; in particular, the generalized Noether theorem is used to obtain symmetries and first integrals.

1. INTRODUCTION

The search for symmetries and first integrals of the long-established two-dimensional Lotka–Volterra (LV) system describing the dynamics of a predator–prey ecosystem has attracted some attention in the nineties (Nutku, 1990; Baumann and Freyberger, 1991; Shentil Velan and Lakshmanan, 1995). Several methods have been used: the Lie method, Painlevè analysis, etc. Some authors (e.g., Baumann and Freyberger, 1991) have expressed regret that we cannot appeal to Noether (first) theorem in examining the symmetries and the first integrals of the model since there is no Lagrangian from which to derive the differential equations at issue.

Another aspect of interest in the literature is the analysis of the Hamiltonian structure(s) of LV systems. The Hamiltonian structure is already known for the two-dimensional case (Nutku, 1990; Baumann and Freyberger, 1991), but to the best of our knowledge a possible underlying Lagrangian structure (leading, perhaps, to this Hamiltonian) has not been investigated. It is the purpose of this paper to show that the two-dimensional LV system has such a structure and consequently it would be of interest to use the methods of analysis of Langrangian mechanics; in particular, in this paper the constructive

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version of Noether theorem given in Cariñena and Fernández-Núñez (1993) is used to find the symmetries and first integrals already known in the literature. The Lagrangian is linear in the velocities and therefore singular (Cariñena *et al.*, 1988; Fernández-Núñez, 1994), although the corresponding dynamics is unique and the Noether theorem works. The one-to-one relation between (generalized) symmetries and first integrals (Cariñena *et al.*, 1992) does not apply because of the singularity of the Lagrangian; we will show how independent (generalized) symmetries lead to the same first integral. A remarkable aspect of the Lagrangian structure we have found is that it is just the already noted Hamiltonian one (Nutku, 1990; Baumann and Freyberger, 1991).

To begin with, we recall that the Euler–Lagrange equations corresponding to a Lagrangian linear in the velocities (Fernández-Núñez, 1994)

$$L(q, \dot{q}) = A_i(q)\dot{q}^i - B(q) \tag{1}$$

constitute a system of first-order differential equations, namely

$$\Omega_{ij}\dot{q}^j - \omega_i = 0 \tag{2}$$

where $\Omega_{ij} = \partial A_j/\partial q^i - \partial A_i/\partial q^j$ and $\omega_i = \partial B/\partial q^i$. That means that the differential forms $\alpha = \Sigma_{i<j}\, \Omega_{ij} dq^i \wedge dq^j$ and $\beta = \omega_i dq^i$ are exact [in fact $\alpha = d(A_i dq^i)$ and $\beta = dB$].

Notice that $\Omega = (\Omega_{ij})$ is skew-symmetric, and consequently only when the number of degrees of freedom is even could Ω be invertible and it will be possible to put the system (2) in normal form $\dot{q}^i = \Omega^{ij}\omega_j = \Omega^{ij}\partial B/\partial q^j$, $\Omega_{ij}\Omega^{jk} = \delta_i^k$. That "regular" case is very important because the dynamics is unique and it is but a (generalized) Hamiltonian system, the Hamiltonian function being the "energy" function $E_L = B$, while the symplectic structure is determined by the matrix Ω. Half of the coordinates are the "coordinates" while the other half correspond to the "momenta."

2. THE LAGRANGIAN STRUCTURE

The 2D LV system, representing the time evolution of a self-regulating predator–prey system, is given by the first-order ordinary differential equations

$$\begin{aligned} \dot{x} &= x(a - y) \\ \dot{y} &= -y(b - x) \end{aligned} \tag{3}$$

where a and b are constant parameters and the dot means time differentiation as usual. We will limit ourselves to the case for which both x and y do not vanish; when $x(0) = 0$ then $x(t) = 0\ \forall t$ and we have a one-dimensional system [the same argument works for $y(0) = 0$]. System (3) can be set up as a Lagrangian one. As (3) is a first-order system, we must consider a Lagrangian linear in the velocities $\dot{x}$, $\dot{y}$.

Our system (3) can be put into the form (2) for $n = 2$ when $x \neq 0$ and $y \neq 0$, namely $\dot{x}/xy - a/y + 1 = 0$, $-\dot{y}/xy - b/x + 1 = 0$, for which

$$\Omega = \begin{pmatrix} 0 & -1/xy \\ 1/xy & \end{pmatrix} \quad \text{and} \quad \omega = \begin{pmatrix} b/x - 1 \\ a/y - 1 \end{pmatrix} \tag{4}$$

Both $\alpha = (-1/xy)dx \wedge dy$ and $\beta = (b/x - 1)dx + (a/y - 1)dy$ are exact; in fact $\alpha = d((\ln|y|/2x)dx - (\ln|x|/2y)dy)$ and $\beta = d(a \ln|y| + b \ln|x| - x - y)$. Consequently, (3) can be considered as the Euler–Langrange equations arising from the Langrangian

$$L = \frac{1}{2}\frac{\ln|y|}{x}\dot{x} - \frac{1}{2}\frac{\ln|x|}{y}\dot{y} - (a \ln|y| + b \ln|x| - x - y) \tag{5}$$

The Lagrangian (5) is linear in the velocities and therefore singular. Although the dynamics corresponding to a singular Lagrangian does not exist nor is unique for any initial state, the regularity of Ω ($x \neq 0$, $y \neq 0$) in our case implies a unique dynamics. The resulting system can be considered as a Hamiltonian one on the manifold $\mathbf{R}^2 - \{x = 0\} - \{y = 0\}$ with symplectic structure Ω. In addition, the Hamiltonian function (i.e., the energy) reads

$$E_L = a \ln|y| + b \ln|x| - x - y \tag{6}$$

3. SYMMETRIES AND FIRST INTEGRALS

Although the Lagrangian (5) is singular, we can use a generalized version of Noether (first) theorem (Cariñena *et al.*, 1992) to find the symmetries and the corresponding first integrals. While the theorem admits a converse in the regular case, for singular L only the direct theorem works. In fact, a given first integral would be associated with independent symmetries, as we will see below.

We recall that a (generalized) symmetry of a Lagrangian L is a vector field $X = X^i(t, q, \dot{q})\partial/\partial q^i$ such that the variation of L along the first prolongation of X is the total time derivative of some function F. We can use this definition in the search of the symmetries of L, looking simultaneously for both the symmetry vector X and the associated function F (Cariñena and Fernández-Núñez, 1993).

First of all, note that our Langrangian (5) is time independent so the energy E_L (6) is a first integral, the corresponding symmetry vector being

$X = \dot{x}\partial/\partial x + \dot{y}\partial/\partial y$ (Cariñena and Fernández-Núñez, 1993). This symmetry X is the vector v_3 obtained in Baumann and Freyberger (1991) for the restrictive case in which $a + b = 0$. As is well known, X represents the invariance under time translation. The energy E_L is a Liapunov function for the LV system (Hirsch and Smale, 1974) and it is used in qualitative analysis of the dynamics.

Let us use the above method to find some symmetries in the particular case when no dependence of the velocities is considered. We have to find a vector

$$X = f(t, x, y)\frac{\partial}{\partial x} + g(t, x, y)\frac{\partial}{\partial y} \tag{7}$$

such that there is a function $F(t, x, y)$ satisfying the equation

$$f\frac{\partial L}{\partial x} + g\frac{\partial L}{\partial y} + \dot{f}\frac{\partial L}{\partial \dot{x}} + \dot{g}\frac{\partial L}{\partial \dot{y}} = F \tag{8}$$

The Noether (first) theorem says that the function $G = F - (f\partial L/\partial \dot{x} + g\partial L/\partial \dot{y})$ is a first integral (Cariñena *et al.*, 1992).

The search for the solutions of (8), both F and the components of the symmetry vector f and g, is a very hard task, but the problem admits a great simplification if we look for solutions f and g having the form

$$\begin{aligned} f(t, x, y) &= \tau(t)\varphi(x, y) \\ g(t, x, y) &= \tau(t)\psi(x, y) \end{aligned} \tag{9}$$

Condition (8) leads to

$$\begin{aligned} \tau(t) &= e^{-kt} \\ \varphi(x, y) &= xy\frac{\partial\phi}{\partial y} \\ \phi(x, y) &= -xy\frac{\partial\phi}{\partial x} \end{aligned} \tag{10}$$

where $\phi(x, y)$ is a solution of the first-order linear PDE

$$x(a - y)\frac{\partial\phi}{\partial x} - y(b - x)\frac{\partial\phi}{\partial y} = k\phi \tag{11}$$

where k is an arbitrary constant. In addition, the function F and the first integral G are

$$F = e^{-kt}\left[y\frac{\partial\phi}{\partial y}\left(\frac{b-x}{k} - \frac{\ln|y|}{2}\right) - x\frac{\partial\phi}{\partial x}\left(\frac{a-y}{k} + \frac{\ln|x|}{2}\right)\right] \text{ when } k \neq 0$$

$$F = -\frac{y}{2}\frac{\partial\phi}{\partial y}\ln|y| - \frac{x}{2}\frac{\partial\phi}{\partial x}\ln|x| + \phi \qquad \text{when} \quad k = 0 \tag{12}$$

$$G = e^{-kt}\,\phi\,(x, y)$$

Consequently, in order to find symmetries of type (9) and their related first integrals the main equation to solve is (10). Using the method of characteristics, we arrive at a transcendent equation, namely the conservation of the energy (6), so that it is very hard to analyze the general solution.

We will content ourselves with examining some particular solutions. For instance, when $k = 0$ the only independent solution is $\phi = a\ln|y| + b\ln|x| - x - y$, i.e., $\phi = E_L$, and the symmetry vector is

$$X = x(a-y)\frac{\partial}{\partial x} - y(b-x)\frac{\partial}{\partial y} \tag{13}$$

Notice that the conservation of the energy corresponds at least to two independent symmetries. As pointed out above, this is not surprising due to the singular character of L. In addition, (13) is the symmetry S_1 found in Shentil Velan and Lakshmanan (1995) when the parameters a and b satisfy the restrictive condition $a + b = 0$.

Solutions of (11) such that $\partial\phi/\partial x = \partial\phi/\partial y$ exist only when $a + b = 0$ and are of the form $\phi(x, y) = (x + y)^{k/a}$. Particular interesting cases are $k = (n+1)a$, $n \neq -1$ being an integer. The associated first integrals

$$G_n = e^{-(n+1)at}(x+y)^{n+1} \tag{14}$$

are essentially the same for $G_n = G_0^{n+1}$, and the corresponding symmetry vectors

$$X_n = (n+1)e^{-(n+1)at}xy(x+y)^n\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) \tag{15}$$

are obviously linearly dependent. The symmetry vector X_0 coincides with the symmetry vectors S_2 in Shentil Velan and Lakshmanan (1995) and v_1 in Baumann and Freyberger (1991), whereas X_1 coincides with S_3 in Shentil Velan and Lakshmanan (1995) and v_2 in Baumann and Freyberger (1991). Incidentally, Shentil Velan and Lakshmann (1995) claim to have found a symmetry quadratic in y [vector S_4 equation (4.3) in their paper]. However, that vector neither satisfies the symmetry criterion [equation (A.6) in Shentil Velan and Lakshmanan (1995)] nor is a Noether symmetry [equation (11) when $a + b = 0$ and $k = a$, this paper].

Solving (11) by the method of separation of variables, i.e., $\phi(x, y) = X(x)Y(y)$, yields no new first integrals. In fact, the method only works when $k = 0$ and we get $\phi = x^b y^a / e^{x+y} = e^{E_L}$.

4. FINAL REMARKS

In this paper we have shown that the 2D Lotka–Volterra system can be analyzed using the methods of Lagrangian dynamics: we find a Lagrangian structure from which the already known symmetries and first integrals derive from Noether theorem. Other (independent) symmetries could be obtained from (11) or by eliminating the restrictive condition (9) (separation of t variable).

Equation (11) can be obtained directly from the definition for the function $G = e^{-kt}\phi(x, y)$ to be a first integral. In this sense, the constructions leading to (11) would seem to be superfluous. This is not the case because we were interested in finding the symmetry vectors also.

As another remark, note that we have chosen the ansatz (7). The theory of generalized symmetries (Cariñena and Fernández-Núñez, 1993) allows symmetry vectors whose components depend on the velocities. But for the linear Lagrangian (5) this possible extension would be reduced to (7) by means of the equations of motion (3). By the same reason, the time-translation symmetry vector is equivalent to the symmetry vector (13), both giving rise to the integral of the energy (6).

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